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Representations of $SO(p+q)$ and $O(p)$ with an Application to the Shapes of $^{8,9}Li$

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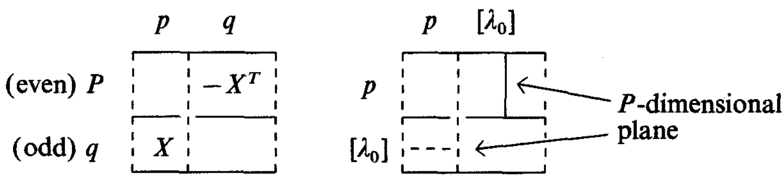
It is easy to show that the symmetry groups governing a system of Z protons and N neutrons are $SO(p+q)$ and $O(p)$, where p, q are related to Z, N and the symmetry groups are transitive on a Grassmann manifold $G_{p,q}$. In this paper the general representations of $SO(p+q)$ and $O(p)$ are found and used to describe the geodesics on $G_{p,q}$ for the nuclear manifolds of the neutron rich-elements $^{8,9}Li$.

1. INTRODUCTION

Some time ago de Wet (1987) showed that $SO(p+q)$ is the correct symmetry group for the odd-A nuclei, while $O(p)$ governs the even nuclei. Here

$$\begin{aligned} p = q &= \tfrac{1}{2}(Z+1)(N+1) && \text{if } Z \text{ or } N \text{ or both are odd} \\ &= (q+1) = \tfrac{1}{2}[(Z+1)(N+1)+1] && \text{if both } Z, N \text{ are even} > 2 \end{aligned} \tag{1.1}$$

A matrix representation (1.11) of $SO(p+q)$ may be written in the supermanifold form



(1.2)

and the group rotates a p-dimensional plane (labeled by the spin state $[\lambda_0]$).

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Since there may be more than one angle between two p planes (Wong, 1967), this means that there may be more than one angular momentum series.

The rows (columns) of (1.2) are labeled by a partition

$$A = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \tag{1.3}$$

of A such that $(\lambda_1 + \lambda_2)$, $(\lambda_2 + \lambda_3)$, $(\lambda_2 + \lambda_4)$ remain constant. Physically $(\lambda_1 + \lambda_2)$, $(\lambda_3 + \lambda_4)$ are respectively the numbers N, Z; while $(\lambda_2 + \lambda_3)$ is the number of nucleons with a given spin and $(\lambda_2 + \lambda_4)$ that number with a given parity. Thus the possible states associated with a given state $[\lambda_1 \lambda_2 \lambda_3 \lambda_4] \equiv [\lambda]$ are

$$\begin{aligned} &\{N00Z; N01(Z-1); \ldots; N0Z0\}; \\ &\{(N-1)10Z; (N-1)11(Z-1), \ldots; (N-1)1Z0\}; \\ &\ldots\{0N0Z; 0N1(Z-1); \ldots; 0NZ0\} \end{aligned} \tag{1.4}$$

These include all the possible spin values of the multiplets $j_1 = N/2$, $j_2 = Z/2$ with the principal state $[\lambda]$ in the middle row. An example for ^{8}Li, ^{8}B is provided by Table I.

There are in fact many principal states

$$\Psi^{(A)} = \sum_{\lambda} C_{[\lambda]} P_{[\lambda]} \tag{1.5}$$

associated with all the possible configurations of A nucleons (de Wet,

Table I. Coherent States of ^{8}Li, ^{8}B

^{8}Li		^{8}B		^{8}Li		^{8}B		$C_{[3203]}$
$\lambda_1\lambda_2\lambda_3\lambda_4$	$\lambda_2\lambda_1\lambda_4\lambda_3$	$\lambda_3\lambda_4\lambda_1\lambda_2$	$\lambda_4\lambda_3\lambda_2\lambda_1$	π_0	s_0	π_0	s_0	$= C_{[3023]}$
S+	−	−	+					
5003	0530	0350	3005	$-2i$	$8i$	$2i$	$8i$	60
5012	0521	1250	2105	$-4i$	$6i$	$4i$	$6i$	20
5021	0512	2150	1205	$-6i$	$4i$	$6i$	$4i$	−20
5030	0503	3050	0305	$-8i$	$2i$	$8i$	$2i$	−60
4103	1430	0341	3014	0	$6i$	0	$6i$	36
4112	1421	1241	2114	$-2i$	$4i$	$2i$	$4i$	12
4121	1412	2141	1214	$-4i$	$2i$	$4i$	$2i$	−12
4130	1403	3041	0314	$-6i$	0	$6i$	0	−36
A3203	2330	0332	3023A	$2i$	$4i$	$-2i$	$4i$	12
3212	2321	1232	2123	0	$2i$	0	$2i$	4
3221	2312	2132	1223	$-2i$	0	$2i$	0	−4
3230	2303	3032	0323	$-4i$	$-2i$	$4i$	$-2i$	−12

1971). Here $C_{[\lambda]}$ is an invariant operator and $P_{[\lambda]}$ is a projection operator consisting of all four possible charge spin states. It satisfies

$$P_{[\lambda]}^2 = P_{[\lambda]}\psi \tag{1.6}$$

so $P_{[\lambda]}/\psi$ is idempotent, but since the parameter ψ is irrelevant, $P_{[\lambda]}$ is a pure or coherent state in the density matrix formulation of quantum mechanics (Biedenharn and Louck, 1981) with the eigenvalues 0, ψ, which implies that only one state $[\lambda]$ can exist at a particular time.

Essentially the analysis consists in finding the tensor products in the enveloping algebra $A(\gamma)$ of the Dirac ring of a self-representation

$$\tfrac{1}{4}\Psi = (iE_4\psi_1 + E_{23}\psi_2 + E_{14}\psi_3 + E_{05}\psi_4)e \tag{1.7}$$

with itself (de Wet, 1971). Here Eddington's E-numbers are related to the 4×4 Dirac matrices by

$$\gamma_\nu = iE_{0\nu}, \quad E_{\mu\nu} = E_{0\mu}E_{0\nu}, \quad E_{\mu\nu}^2 = -1, \quad E_{\mu\nu} = -E_{\nu\mu}, \quad \mu < \nu = 1, \ldots, 5$$

and the commuting operators E_{23}, E_{14}, and E_{05} are, respectively, infinitesimal rotations in 3-space, 4-space, and isospace that correspond to the spin σ, parity π, and charge T_3 carried by a single nucleon. The parameters ψ_2, ψ_3, ψ_4 are half-angles of rotation and e is a primitive idempotent; E_4 is a unit matrix.

A rotation through 180° about x will change spin up to spin down and if this is followed by a rotation of 180° about t, then x can go to $-x$ without inverting time but causing a space inversion, i.e., a left-handed coordinate system or parity reversal ($E_{14} \to -E_{14}$).

The basis elements of $A(\lambda)$ are the $4^A \times 4^A$ matrices

$$E_{\mu\nu}^l = E_4 \otimes \cdots \otimes E_4 \otimes E_{\mu\nu} \otimes E_4 \otimes \cdots \otimes E_4 \tag{1.8a}$$

with $E_{\mu\nu}$ in the lth position. The elements $E_{\mu\nu}^l$, $E_{\mu\nu}^{l+1}$ commute and $A(\gamma)$ is found to have the following generators:

$$\Gamma_\nu^{(A)} = \tfrac{1}{2}(E_{0\nu}^1 + E_{0\nu}^2 + \cdots + E_{0\nu}^A), \qquad \nu = 1, \ldots, 5 \tag{1.8b}$$

$$\sigma_{\mu\nu}^{(A)} = [\Gamma_\mu^{(A)}, \Gamma_\nu^{(A)}] = \tfrac{1}{2}(E_{\mu\nu}^1 + E_{\mu\nu}^2 + \cdots + E_{\mu\nu}^A) \tag{1.8c}$$

$$\eta_\nu^{(A)} = E_{0\nu} \otimes \cdots \otimes E_{0\nu} = E_{0\nu}^1 E_{0\nu}^2 \cdots E_{0\nu}^A \tag{1.8d}$$

$$\eta_{\mu\nu}^{(A)} = \eta_\mu^{(A)}\eta_\nu^{(A)} = E_{\mu\nu}^1 E_{\mu\nu}^2 \cdots E_{\mu\nu}^A, \qquad \mu < \nu = 1, \ldots, 5 \tag{1.8e}$$

These algebras were called de Broglie algebras by Boerner (1963) and the Kemmer algebra corresponds to the particular case $A = 2$ (Kemmer, 1943). Then

$$\begin{aligned} i^A P_{[\lambda]} = (&i^A \psi_1^{\lambda_1}\psi_2^{\lambda_2}\psi_3^{\lambda_3}\psi_4^{\lambda_4} + \eta_{23}^{(A)}\psi_2^{\lambda_1}\psi_1^{\lambda_2}\psi_4^{\lambda_3}\psi_3^{\lambda_4} \\ &+ \eta_{14}^{(A)}\psi_3^{\lambda_1}\lambda_4^{\lambda_2}\psi_1^{\lambda_3}\psi_2^{\lambda_4} + \eta_5^{(A)}\psi_4^{\lambda_1}\psi_3^{\lambda_2}\psi_2^{\lambda_3}\psi_1^{\lambda_4})\epsilon_A \end{aligned} \tag{1.9}$$

where $\epsilon_A = e \otimes \cdots \otimes e = e^1 e^2 \cdots e^A$ is a primitive idempotent in $A(\gamma)$ and $\psi \equiv \psi_1 \psi_2 \psi_3 \psi_4$.

Equation (1.9) is a generalization in many-particle space of the fundamental representation (1.7) and by examination we may confirm the canonical labeling introduced above. The first two terms are characterized by the same values of Z, N but opposite spins because $(\lambda_2 + \lambda_3)$ is replaced by $(\lambda_1 + \lambda_4)$. In the third and fourth terms $(\lambda_3 + \lambda_4)$ has replaced $(\lambda_1 + \lambda_2)$, so charge has been reversed, but again there are two possible spin states in the mirror nucleus. These changes lead automatically to a parity reversal in the first and second pairs and it may easily be shown that the choice of $(\lambda_2 + \lambda_3)$ as the number of nucleons with a negative spin is in agreement with the labeling of the rows of the outer product $|j_1 m_1\rangle \otimes |j_2 m_2\rangle$ of a system of protons and neutrons described by (1.4).

However, many configurations in the tensor product are the same up to a combination that includes every possible exchange of spin, parity, and charge between nucleons such that the net quantum numbers remain the same for that state $[\lambda]$. This is equivalent to constructing the quotient space $M = P/H$, where P is the 4^A-dimensional configuration space, with coordinates x_i carrying $A(\gamma)$ and H is the permutation group. Thus, if

$$N_{[\lambda]} = A!/\lambda_1!\lambda_2!\lambda_3!\lambda_4! \tag{1.10}$$

is the number of combinations of the state $[\lambda]$, we may regard the $N_{[\lambda]}$ phase changes as a fiber over the point ξ_i in M so that the model is invariant under phase transformations.

Then by choosing 4×4 matrix representations for E_{23}, E_{14}, E_{05} and constructing the fibers it may be shown (de Wet, 1973, 1987) that the base space M decomposes beautifully into subspaces constituting an isobaric multiplet, and moreover that the spin angular momentum matrices in a given member of the multiplet are just those of Biedenharn and Louck (1981) for a coupled system of protons and neutrons, namely

$$\sigma_i = E_N \otimes {}^P\Gamma_i + {}^N\Gamma_i \otimes E_P, \qquad i = 1, 2, 3 \tag{1.11a}$$

where ${}^P\Gamma_i$ and ${}^N\Gamma_i$ are $(P+1)$- and $(N+1)$-dimensional Lie operators of $SO(3)$; E_P and E_N are $(P+1)$ and $(N+1)$ unit matrices; and $\sigma_i \equiv \sigma_{jk}^{(A)}$ of (1.8c), so that the spins of the A nucleons are indeed additive.

Similarly the infinitesimal rotations in 4-space are

$$\pi_i \equiv \sigma_{i4}^{(A)} = E_N \otimes {}^P\Gamma_i - {}^N\Gamma_i \otimes E_P \tag{1.11b}$$

and σ_i and π_i are the infinitesimal operators of the 4-dimensional rotation group $O_4 \supset SO(p+q)$, so we have found a supermultiplet augmented by the invariant operator

$$C_{[\lambda]} = i^{\lambda_1} C(E_{23}^1 \cdots E_{23}^{\lambda_2}\, E_{14}^{\lambda_2+1} \cdots E_{14}^{\lambda_2+\lambda_3}\, E_{05}^{\lambda_2+\lambda_3+1} \cdots E_{05}^{A-\lambda_1}) \tag{1.12}$$

where C denotes summation over the $N_{[\lambda]}$ combinations of the basis elements contained in the bracket. Special cases are

$$C_{[(A-1)100]} = 2_i^{A-1}\sigma_1, \qquad C_{[(A-1)010]} = 2_i^{A-1}\pi_1, \qquad C_{[(A-1)001]} = 2i^{A-1}\Gamma_5^{(A)} \tag{1.13}$$

and by rearranging rows and columns we can rewrite σ_1, π_1 in the form (1.2) if A is odd.

However, it is also possible to choose a *CP-invariant operator* $C_{[\Lambda]}$ from among the principal operators $C_{[\lambda]}$ of (1.5), and following the work of Arnold (1989), the nuclear dynamics is considered to be governed by $C_{[\Lambda]}$, which is an exchange operator and a bilinear form, or measure, on the nuclear manifold M. This complements the ideas of Gilmore and Draayer (1985) and Barut and Raczka (1977), who also replaced the Hamiltonian by an invariant operator. Thus the theory of nuclear structure reduces to the purely mathematical problem of studying the associated hyperspherical functions in the middle column of $\exp(C_{[\Lambda]}\theta)$, which are at the same time representations of $SO(p+q)$ or $O(p)$. Actually $C_{[\Lambda]}$ is reducible, but if A is odd one can find an irreducible subspace μ of $SO(p+q)$ containing the fundamental state $[\Lambda]$ so that the irreducible representations of $SO(p+q)$ are provided by exponentiation of $(\mu\theta)$ and come from a horizontal subspace m of the tangent space to $G_{p,q}$. But if A is even, then μ is a subspace of $O(p)$ which belongs to the vertical subspace h of the tangent space.

In the next section a general solution will be given to the problem of exponentiating the infinitesimal operators $o(P)$ and $so(p+q)$, which have real and complex eigenvalues, respectively. Thus there are two classes: (1) real eigenvalues, $\mu \in h$, A even; (2) imaginary eigenvalues, $\mu \in m$, A odd. Then in the final section the theorem will be used to find the structure of the neutron-rich nuclei ^{8,9}Li, which have recently been studied (Blank *et al.*, 1991).

2. IRREDUCIBLE REPRESENTATIONS OF $SO(p+q)$, $O(p)$

As stated at the end of the last section, we will be concerned with two cases according to whether the irreducible subspace μ belongs to the vertical or horizontal subspaces of the canonical decomposition $g = h + m$ of the Lie algebras $o(p)$ and $so(p+q)$. Beginning with class 1, μ symmetric$\in h$, we can always reduce the eigenvalues to the positive canonical form $0, 1, \lambda_2 \cdots \lambda_n$ by subtracting a constant λ_t to translate the spectrum and then dividing by a factor λ_f so that one eigenvalue of the translated spectrum has the value unity. This follows because if $AX = \lambda X$, then

$$(A - \lambda_t)X = (\lambda - \lambda_t)X$$

It will be shown in Section 3 that exponentiation of the translated or canonical spectrum leads to a factor $e^{i\lambda_t\theta}$ that is responsible for vibrational modes, while λ_f may be absorbed in θ and does not change the shape of the geodesics although there is a frequency change. Once in possession of the canonical spectrum it is always possible to use the characteristic equation

$$\mu(\mu-1)(\mu-\lambda_2)\cdots(\mu-\lambda_n)=0 \tag{2.1a}$$

to express μ^{n+m} in terms of $\mu, \mu^2, \ldots, \mu^n$ and thus to write

$$e^{i\mu\rho}=1+\sum_{j=1,2,\ldots}^{n}\mu^j\Sigma_j(\theta)+i\sum_{j=1,2,\ldots}^{n}\mu^jS_j(\theta) \tag{2.1b}$$

where $\Sigma_j(\theta)$ is a series in even powers of θ with alternating sign, while $S_j(\theta)$ is a series in odd powers of θ.

If μ, skew-symmetric ϵm, is a class 2 matrix, then the eigenvalues are imaginary and the characteristic equation is

$$\mu(\mu^2+1)(\mu^2+\lambda_2^2)\cdots(\mu^2+\lambda_n^2)=0 \tag{2.2a}$$

and

$$e^{\mu\theta}=1+\sum_{j=2,4,6,\ldots}^{2n}\mu^j\Sigma_j(\theta)+\sum_{j=1,3,5,\ldots}^{2n-1}\mu^jS_j(\theta) \tag{2.2b}$$

Before stating the central theorem of this contribution, we note that the sum of the coefficients of the characteristic equations (2.1a), (2.2a) is zero. This follows trivially by setting $\mu=1$ in (2.1a), and $\mu=i$ in (2.2a), and ensures that the exponential series have rotational properties. To see this we will prove two lemmas.

Lemma 1. If the characteristic equation (2.2a) is written

$$\mu^{2n+1}=-C_{2n-1}\mu^{2n-1}-C_{2n-3}\mu^{2n-3}-\cdots-C_1\mu \tag{2.2c}$$

where

$$C_{2n-1}-C_{2n-3}+C_{2n-5}-\cdots\pm C_1=1 \tag{2.2d}$$

then the sum of the coefficients of the characteristic equation

$$\begin{aligned}\mu^{2n+3}&=-C_{2n-1}\mu^{2n+1}-C_{2n-3}\mu^{2n-1}-C_{2n-5}\mu^{2n-3}-\cdots-C_1\mu^3\\&=C_{2n-1}(C_{2n-1}\mu^{2n-1}+C_{2n-3}\mu^{2n-3}+\cdots+C_1\mu)\\&\quad-C_{2n-3}\mu^{2n-1}-\cdots-C_1u^3\end{aligned} \tag{2.2e}$$

with alternating signs, will also be unity.

This follows directly from (2.2d) and by induction the lemma will be true for all odd powers of μ. It will also be true for even powers where

(2.2c) is replaced by

$$\mu^{2n+2} = -C_{2n-1}\mu^{2n} - C_{2n-3}\mu^{2n-2} \cdots - C_1\mu^2 \tag{2.2f}$$

and for all the powers of μ derived from successive substitution of the class 1 characteristic equation (2.1a). We will call the coefficients of (2.2e) and its successors daughters of (2.2d).

Lemma 2. For the class 2 functions;

$$\frac{1}{i}\sum_{j=1,3,5,\ldots}^{2n-1} i^j S_j(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots = \sin\theta \tag{2.3a}$$

when j is odd and

$$1 + \sum_{j=2,4,6,\ldots}^{2n} i^j \Sigma_j(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots = \cos\theta \tag{2.3b}$$

when j is even.

This follows from Lemma 1 because the sum of the coefficients of $\theta^k/k!$ will be unity by virtue of (2.2d) and its daughters, since (2.2b) associates a daughter with just those powers of μ appearing in (2.2c), (2.2f), for each k.

Similarly, in case 1,

$$\sum_{j=1,2,\ldots}^{n} S_j(\theta) = \sin\theta \tag{2.4a}$$

$$1 + \sum_{j=1,2,\ldots}^{n} \Sigma_j(\theta) = \cos\theta \tag{2.4b}$$

So putting $\mu = 1$ in (2.1b) and $\mu = i$ in (2.2b), we recover the basic circular relation

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2.5}$$

which demonstrates the required rotational properties. As will be seen in the next section, we will only be interested in the functions $S_j(\theta)$ of class 2 and $\Sigma_j(\theta)$ of class 1 [the remainder may be obtained by integration and (2.3), (2.4)] and we can now state the main contribution of this paper.

Theorem. The rotational functions of class 1 are

$$\Sigma_j(\theta) = b_{j0} + b_{j1}\cos\theta + b_{j2}\cos\lambda_2\theta + \cdots + b_{jn}\cos\lambda_n\theta \tag{2.6}$$

where $j = 1, 2, \ldots, n$. In case 2 they are

$$i^{j-1}S_j(\theta) = a_{j1}\sin\theta + a_{j2}\sin\lambda_2\theta + \cdots + a_{jn}\sin\lambda_n\theta \tag{2.7}$$

where j is odd, and $\lambda_2, \ldots, \lambda_n$ are positive.

Proof. We will begin by proving (2.7). By (2.2b)

$$\frac{d}{d\theta} e^{\mu\theta}\bigg|_{\theta=0}$$
$$= \mu^2 \frac{d}{d\theta}\Sigma_2(\theta)\bigg|_{\theta=0} + \mu^4 \frac{d}{d\theta}\Sigma_4(\theta)\bigg|_{\theta=0} + \cdots + \mu^{2n}\frac{d}{d\theta}\Sigma_{2n}(\theta)\bigg|_{\theta=0}$$
$$+ \mu\frac{d}{d\theta}S_1(\theta)\bigg|_{\theta=0} + \mu^3\frac{d}{d\theta}S_3(\theta)\bigg|_{\theta=0} + \cdots + \mu^{2n-1}\frac{d}{d\theta}S_{2n-1}(\theta)\bigg|_{\theta=0} = \mu \tag{a}$$

and only the $dS_1(\theta)/d\theta$ term survives, so we find from (2.7) that

$$a_{11} + \sum_{j=2,3,\ldots}^{n} \lambda_j a_{ij} = 1$$
$$a_{31} + \sum_{j=2,3,\ldots}^{n} \lambda_j a_{3j} = \cdots = a_{2n-1,1} + \sum_{j=2,3,\ldots}^{n} \lambda_j a_{2n-1,j} = 0 \tag{b}$$

Also (2.3a) says that if (2.7) is a solution, then

$$A_1 = \sum_{j=1,3,5,\ldots}^{2n-1} a_{j1} = 1$$
$$\left(A_2 = \sum_{j=1,3,5,\ldots}^{2n-1} a_{j2}\right) = \left(A_3 = \sum_{j=1,3,\ldots,}^{2n-1} a_{j3}\right) = \cdots = \left(A_n = \sum_{j=1,3,\ldots}^{2n-1} a_{jn}\right) = 0 \tag{c}$$

and the proof reduces to confirming the relations (b), (c) by comparing the series expansion of (2.7) with the $i^{j-1}S_j(\theta)$ of (2.2b). We easily confirm that (b) follows from the fact that the coefficient of θ is unity in $S_1(\theta)$ and zero in $S_3(\theta), \ldots S_{2n-1}(\theta)$ [see (3.17)]. In general the coefficient of $\theta^k/k!$ in $i^{j-1}S_j(\theta)$ is

$$a_{j1} + \sum_{i=2,3,\ldots}^{n} \lambda_i^k a_{ji} \tag{d}$$

and adding

$$A_1 + \sum_{i=2,3,\ldots}^{n} \lambda_i^k A_i = 1 \tag{e}$$

which is satisfied by (c), since the λ_i are arbitrary and nonzero, this completes the first half of the proof.

To prove (2.6), we note that in the case of the cosine expansion all the terms in $d(e^{i\mu\theta})/d\theta|_{\theta=0}$ vanish, so we do not need (b), while (2.4b) implies that

$$B_0 = \sum_{i=1,2,\ldots}^{n} \sum_{j=1,2,\ldots}^{n} b_{ij} = 0 \tag{f}$$

$$B_1 = \sum_{j=1,2,\ldots}^{n} b_{j1} = 1, \qquad \left(B_2 = \sum_{j=1,2,\ldots}^{n} b_{j2}\right) = \cdots \left(B_k = \sum_{j=1,2,\ldots}^{n} b_{jk}\right) = 0 \tag{g}$$

which we can confirm by expanding (2.6) and comparing with $\Sigma_i(\theta)$ of (2.1b). The coefficient of unity is (f), while the coefficient of $\theta^k/k!$ yields

$$B_1 + \sum_{j=2,3,\ldots}^{n} \lambda_j^k B_j = 1$$

which is satisfied by (g) [see (3.8)]. This completes the proof.

In case 2, (2.2e) and (2.2f) show that the even and odd powers $\theta^{2n+1}/(2n+1)!$ and $\theta^{2n}/2n!$ will have the same coefficient C_{2n-1}, which means that $\Sigma_j(\theta)$ of (2.2b) is also a cosine series that can be found from (2.7) by integration.

The single-value representations of $SO(3)$ have infinitesimal operators that can be written in the form (1.13) (de Wet, 1987) and it will be instructive to show how the theorem holds for this simple case. If we take $j = 2$ as an example, then the characteristic equation

$$0 = \mu(\mu^2 + 1)(\mu^2 + 4) \Rightarrow \mu^5 = -5\mu^3 - 4\mu$$

leads to

$$\begin{aligned} e^{\mu\theta} &= \mu\left(\theta - 4\frac{\theta^5}{5!} + 20\frac{\theta^7}{7!} - 84\frac{\theta^9}{9!} + \cdots\right) \\ &\quad + \mu^3\left(\frac{\theta^3}{3!} - 5\frac{\theta^5}{5!} + 21\frac{\theta^7}{7!} - 85\frac{\theta^9}{9!} + \cdots\right) \\ &\quad + \mu^2\left(\frac{\theta^2}{2!} - 4\frac{\theta^6}{6!} + 20\frac{\theta^8}{8!} - 84\frac{\theta^{10}}{10!} + \cdots\right) \\ &\quad + \mu^4\left(\frac{\theta^4}{4!} - 5\frac{\theta^6}{6!} + 21\frac{\theta^8}{8!} - 85\frac{\theta^{10}}{10!} + \cdots\right) \\ &= 1 + \mu S_1(\theta) + \mu^3 S_3(\theta) + \mu^2\Sigma_2(\theta) + \mu^+ \Sigma_4(\theta) \end{aligned}$$

Then

$$\frac{1}{i}\sum_{j=1,3} i^j S_j(\theta) = S_1(\theta) - S_3(\theta) = \sin\theta$$

and

$$1 + \sum_{j=2,4} i^j \Sigma_j(\theta) = 1 - \Sigma_2(\theta) + \Sigma_4(\theta) = \cos\theta$$

Now let

$$S_1(\theta) = a_{11} \sin \theta + a_{12} \sin 2\theta$$
$$= (a_{11} + 2a_{12})\theta - (a_{11} + 8a_{12}) \frac{\theta^3}{3!} + (a_{11} + 32a_{12}) \frac{\theta^5}{5!}$$
$$= \theta - 4 \frac{\theta^5}{5!}$$

which has the solution

$$a_{11} = \frac{4}{3}, \qquad a_{12} = -\frac{1}{6}$$

Similarly, if

$$-S_3(\theta) = a_{31} \sin \theta + a_{32} \sin 2\theta; \qquad a_{31} = -\frac{1}{3}, \qquad a_{32} = \frac{1}{6}$$

and

$$S_1(\theta) = \frac{4}{3} \sin \theta - \frac{1}{6} \sin 2\theta, \qquad S_3(\theta) = \frac{1}{3} \sin \theta - \frac{1}{6} \sin 2\theta \tag{2.8}$$

it is easily verified that $\mu S_1(\theta) + \mu^3 S_3(\theta)$ agrees with the alternative elements of a representation $D^{(2)}_{m'm}(\pi/2, \theta, \pi)$ of $SO(3)$ and we have succeeded in finding a series solution in terms of arguments of the eigenvalues. In this case it is possible to express this solution in terms of a single angle θ, but this is not in general true for $SO(p+q)$, where more than one angle may be needed to define the relative position of two p planes (Wong, 1967). The remaining elements of $D^{(2)}_{m'm}(\pi/2, \theta, \pi)$ are given by the "cosine" expansion

$$1 + \mu^2 \Sigma_2(\theta) + \mu^4 \Sigma_4(\theta)$$

where

$$\Sigma_2(\theta) = \frac{5}{4} - \frac{4}{3} \cos \theta + \frac{1}{12} \cos 2\theta, \qquad \Sigma_4(\theta) = \frac{1}{4} - \frac{1}{3} \cos \theta + \frac{1}{12} \cos 2\theta$$

Finally, because the arguments depend on the eigenvalues, the theorem is independent of the set of canonical eigenvalues chosen as long as one value is unity or i, and $\mu\theta$ remains constant. Thus, if $\mu \to \mu/n$ (i.e., $\lambda \to \lambda/n$), then $\theta \to n\theta$, which will not change the angular dependence of the geodesics since we can simply replace $n\theta$ by φ and use the new parameter φ.

3. THE NEUTRON-RICH ELEMENTS LITHIUM-8,9

In this section we will begin by finding the CP-invariant operators $C_{[\Lambda]}$ determined by (1.12), then calculate their eigenvalues by using a matrix representation or directly by means of (1.4) and the canonical labeling. We will then be in a position to find the wave functions from (2.1b), (2.6) in the case of ^{8}Li or (2.2b), (2.7) in the case of ^{9}Li. According to a theorem by Kobayashi and Nomizu (1969), this enables us to plot the geodesics on the nuclear manifolds of ^{8,9}Li and if we assume that these geodesics are the paths of mesons holding the nucleus together by exchange forces, then it is possible to find the configuration of protons and neutrons and consequently the nuclear shapes without making any assumption whatsoever about the actual constitution of a nucleon. In accord with experiment (Blank *et al.*, 1991), a nuclear halo is found for ^{8}Li (see Fig. 1).

We can replace (1.12) by

$$C_{[\Lambda]} = i^{\Lambda_1}\sigma_0^{\Lambda_2}\pi_0^{\Lambda_3}T_0^{\Lambda_4} - \sum_{\lambda} i^{\lambda_1}\sigma_0^{\lambda_2}\pi_0^{\lambda_3}T_0^{\lambda_4} \tag{3.1}$$

where

$$\sigma_0 \equiv 2\sigma_{23}^{(A)} = [E_{23}^1 + \cdots + E_{23}^A), \qquad \pi_0 \equiv 2\sigma_{14}^{(A)} = (E_{14}^1 + \cdots + E_{14}^A)$$
$$T_0 = 2\Gamma_5^{(A)} = (E_{05}^1 + \cdots + E_{05}^A) \tag{3.2a}$$

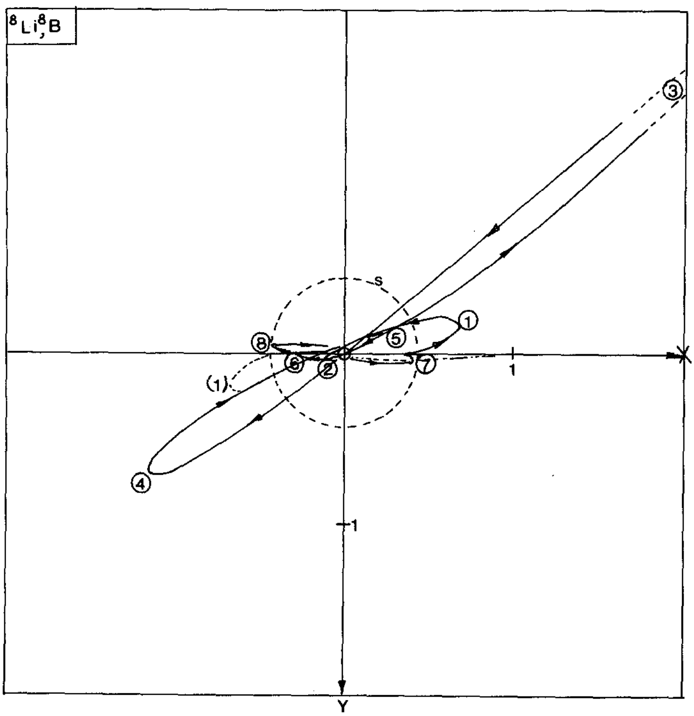

Fig. 1. Geodesics on the manifold of ^{8}Li.

which are related to the real quantum numbers s, p, and $T_3 = \frac{1}{2}(Z - N)$ of spin, parity, and charge by

$$\sigma_0 = 2is, \qquad \pi_0 = 2ip, \qquad T_0 = 2iT_3 \tag{3.2b}$$

and show how the quantum numbers of individual nucleons are additive.

The summation contains all those terms arising from repeated indices $E^j_{23}E^j_{23}$, $E^j_{23}E^j_{14}$, $E^j_{23}E^j_{05}, \ldots$ that yield a single term according to the multiplication table

	E^j_{23}	E^j_{14}	E^j_{05}
E^j_{23}	i^2	iE^j_{05}	iE^j_{14}
E^j_{14}	iE^j_{05}	i^2	iE^j_{23}
E^j_{05}	iE^j_{14}	iE^j_{23}	i^2

(3.3)

Ultimately (3.1) will be expressed in the bilinear form $\Lambda(\sigma_0, \pi_0)$ because for each nucleus, T_0 is just a diagonal matrix equal to $i(Z - N)$. An elementary application of (3.1) is

$$\sigma_0 T_0 = P(E^i_{23}E^j_{05}) + i\pi_0 \tag{3.4a}$$

where P donates summation over the $A!/(A - n)!$ permutations of the bracketted generators. Then

$$C_{[(A-2)101]/i(A-2)} = P(E^i_{23}E^j_{05}) = \sigma_0 T_0 - i\pi_0 \tag{3.4b}$$

In the case of ^{8}Li let us now 'add' another nucleon by multiplying (3.4a) by $\sigma_0 = (E^1_{23} + \cdots + E^8_{23})$ to obtain

$$\sigma_0^2 T_0 = P(E^i_{23}E^j_{23}E^k_{05}) + 2iP(E^i_{23}E^j_{14}) + 8i^2 T_0 \tag{3.5a}$$

and continue the process until

$$\begin{aligned} \sigma_0^2 T_0^3 = {} & P(E^i_{23}E^j_{23}E^k_{05}E^l_{05}E^m_{05}) + 6iP(E^i_{23}E^j_{14}E^k_{05}E^l_{05}) + i^2\{22P(E^i_{23}E^j_{23}E^k_{05}) \\ & + 6P(E^i_{14}E^j_{14}E^k_{05}) + 8P(E^i_{05}E^j_{05}E^k_{05})\} + 44i^3P(E^i_{23}E^j_{14}) + 176i^4T_0 \end{aligned} \tag{3.5b}$$

Now each term in (3.5b) must be expressed in powers of σ_0, π_0, T_0 by using relationships similar to (3.4b) and we can write

$$C_{[3203]} = \frac{i^3}{2!3!} P(E^i_{23}E^j_{23}E^k_{05}E^l_{05}E^m_{05}) = \pi_0^2 - \sigma_0^2 \tag{3.6a}$$

after substituting for $T_0 = i(Z - N) = -2i$. In the case of the mirror nucleus ^{8}B, $T_0 = 2i$ and

$$C_{[3023]} = C_{[3203]} = \pi_0^2 - \sigma_0^2 \tag{3.6b}$$

Thus we have found CP symmetry because $T_3 \to -T_3$ is accompanied by $\pi_0 \to -\pi_0$. In general the fundamental states $[\Lambda]$ need not be ground states but must be invariant under the interchange $\Lambda_1\Lambda_2 \leftrightarrow \Lambda_3\Lambda_4$ if $T_3 = 0$. In the case of the mirror nuclei, $\Lambda_1 = \Lambda_4$, because otherwise $C_{[\Lambda]}$ will not be CP-symmetric under the interchange $\Lambda_1\Lambda_2\Lambda_3\Lambda_4 \leftrightarrow \Lambda_4\Lambda_3\Lambda_2\Lambda_1$. This considerably restricts the number of isotopes; for example, CP invariance is broken in the case of the unbound nucleus ^{10}Li. The number of distinct permutations in (3.6a) is less by the factor $1/\Lambda_2!\Lambda_3!\Lambda_4!$ than the combinations of (1.12) because terms such as $E^i_{14}E^j_{14} = E^j_{14}E^i_{14}$ must not be counted twice.

Table I is an evaluation of (3.6a), (3.6b) for the coherent states of ^{8}Li, ^{8}B, where we have assumed that $(\lambda_2 + \lambda_3)$ is the number of states with a negative spin and $(\lambda_2 + \lambda_4)$ the number with a positive parity.

Thus it is possible to find the eigenvalues of $C_{[\Lambda]}$ without any matrix representation, although we shall need a representation of μ in order to use (2.1b). The state labeling is confirmed by the fact that the eigenvalues are the same in both cases. The first and fourth columns $[\lambda_1\lambda_2\lambda_3\lambda_4]$ and $[\lambda_4\lambda_3\lambda_2\lambda_1]$ have been used to find σ_0 and π_0. If the second and third columns had been used, the spins and parities would have changed sign, but $C_{[\Lambda]}$ is invariant under changes in sign of σ_0 and π_0.

A matrix representation for σ_0 and π_0 is provided by (1.11), which we shall write

$$\sigma_0 = E_5 \otimes \gamma_3 + \gamma_5 \otimes E_3, \qquad \pi_0 = E_5 \otimes \gamma_3 - \gamma_5 \otimes E_3$$

so that

$$C_{[3203]} = -4\gamma_5 \otimes \gamma_3, \qquad C_{[3023]} = -4\gamma_3 \otimes \gamma_5 \tag{3.7}$$

Here

$$\langle jm+1|2\gamma_k|jm\rangle = +[(j-m)(j+m+1)]^{1/2}$$
$$\langle jm-1|2\gamma_k|jm\rangle = -[(j+m)(j-m+1)]^{1/2}, \qquad k = 3, 5$$

is a Lie operator of $SO(3)$ and the matrix representations of $C_{[3203]}$ and $C_{[3023]}$ are identical up to a rearrangement of rows and columns. They are reducible and we must choose that submatrix A with an eigenvalue 12 which from Table I corresponds to $[\Lambda]$. The complete set is $\{60; 12; 12; -4; -20; -36\}$, or adding $\lambda_t = 36$ and dividing by 16, $\{0; 1; 2; 3; 3; 6\}$. Then the characteristic equation

$$\mu(\mu-1)(\mu-2)(\mu-3)(\mu-6) = 0, \qquad \mu = \frac{A}{16} + \frac{9}{4}E_6$$

leads to the real part

$$
\begin{aligned}
e^{i\mu\theta} = 1 &+ \mu\left(432\frac{\theta^6}{6!} - 24192\frac{\theta^8}{8!} + 970704\frac{\theta^{10}}{10!} - \cdots\right) \\
&+ \mu^2\left(-\frac{\theta^2}{2!} - 828\frac{\theta^6}{6!} + 44892\frac{\theta^8}{8!} - 1785420\frac{\theta^{10}}{10!} + \cdots\right) \\
&+ \mu^2\left(492\frac{\theta^6}{6!} - 25032\frac{\theta^8}{8!} + 979524\frac{\theta^{10}}{10!}\cdots\right) \\
&+ \mu^4\left(\frac{\theta^4}{4!} - 97\frac{\theta^6}{6!} + 4333\frac{\theta^8}{8!} - 164809\frac{\theta^{10}}{10!} + \cdots\right)
\end{aligned}
\tag{3.8}
$$

which must be used to find the coefficients b_{ij} of (2.6) by comparison with an expansion of the cosine functions up to $\theta^8/8!$. We find

$$\Sigma_1 = -2 + \frac{18}{5}\cos\theta - \frac{9}{4}\cos 2\theta + \frac{2}{3}\cos 3\theta - \frac{1}{60}\cos 6\theta \tag{3.9a}$$

$$\Sigma_2 = \frac{47}{36} - \frac{18}{5}\cos\theta + \frac{27}{8}\cos 2\theta - \frac{10}{9}\cos 3\theta + \frac{11}{360}\cos 6\theta \tag{3.9b}$$

$$\Sigma_3 = -\frac{1}{3} + \frac{11}{10}\cos\theta - \frac{5}{4}\cos 2\theta + \frac{1}{2}\cos 3\theta - \frac{1}{60}\cos 6\theta \tag{3.9c}$$

$$\Sigma_4 = \frac{1}{36} - \frac{1}{10}\cos\theta + \frac{1}{8}\cos 2\theta - \frac{1}{18}\cos 3\theta + \frac{1}{360}\cos 6\theta \tag{3.9d}$$

and can confirm (2.4b) directly as well as the fact that (3.9) also yields the correct coefficients for $\theta^{10}/10!$. These functions are not yet harmonics because we still have to multiply by the powers of μ obtained from the symmetric matrix

	[1403]	[1421]	[3203] $=[\Lambda]$	[3221]	[5003]	[5021]	$[\lambda_1\lambda_2\lambda_3\lambda_4]$	$C_{[\Lambda]}$
	0	0	0	$-\sqrt{24}$	0	$\sqrt{15}$	[4130]	$-36Y$
	0	0	$-\sqrt{24}$	$2\sqrt{8}$	$\sqrt{15}$	$-2\sqrt{5}$	[4112]	12
$A = 16\mu - 36E_5$	0	$-\sqrt{24}$	0	$3\sqrt{3}$	0	0	[2330]	$12X$
$= -4\times$	$-\sqrt{24}$	$2\sqrt{8}$	$3\sqrt{3}$	-6	0	0	[2312]	-4
	0	$\sqrt{15}$	0	0	0	0	[0530]	60
	$\sqrt{15}$	$-2\sqrt{5}$	0	0	0	0	[0512]	-20

(3.10)

In this particular case the state labeling is not disturbed by (3.7) so it is easy to find the subspace with the labels shown, but this is not generally the case when the fundamental state $[\Lambda]$ has to be found by inspection once

A has been symmetrized by the interchange of rows and columns. If the orthogonal state Y is chosen adjacent to X from the $[\Lambda] = [3203]$ column, then using

$$e^{i(A/16)\theta} = e^{i\mu\theta} e^{-i(9/4)\theta} \tag{3.11}$$

we obtain that the X, Y harmonics of A are finally

$$X = [2330] = \frac{1}{16}(9.875 - 15 \cos\theta + 17.0625 \cos 2\theta + 2.5 \cos 3\theta$$

$$+ 1.5625 \cos 6\theta) \cos\frac{9}{4}\theta \tag{3.12a}$$

$$Y = [4130] = \frac{\sqrt{8}}{16}(2.375 - 4.725 \cos\theta + 3.375 \cos 2\theta - 1.375 \cos 3\theta$$

$$+ 0.350 \cos 6\theta) \cos\frac{9}{4}\theta \tag{3.12b}$$

The factor of 1/16 on the left hand side of (3.11) will not change the shape of the geodesics plotted in Fig. 1 which clearly show a meson path to a "halo" nucleon 3. Only one cycle of $3 \times 360°$ is shown. During the next cycle the nucleons rotate through 180° and there is complete circular motion of the whole configuration. In this case there are no oscillations of the orbits, but in the case of ^{7}Li the factor $e^{i\lambda_i\theta}$ causes a vibration. Also the fact that $\sqrt{8}$ is double-valued means that another set of geodesics in the XY plane can be obtained by reflection.

Turning now to ^{9}Li and ^{9}C, we find CP-invariant operators, respectively,

$$C_{[3303]} = (1/6)(\sigma_0^3 + \pi_0^3) + (3/2)(\sigma_0\pi_0^2 + \sigma_0^2\pi_0) + (17/3)(\sigma_0 + \pi_0) \tag{3.13a}$$

$$C_{[3033]} = (1/6)(\sigma_0^3 - \pi_0^3) + (3/2)(\sigma_0\pi_0^2 - \sigma_0^2\pi_0) + (17/3)(\sigma_0 - \pi_0) \tag{3.13b}$$

The subspace of the matrix representation containing [3303] is

$$M = \frac{2}{3} \times \left[\begin{array}{c|c} 0 & A \\ \hline -A & 0 \end{array}\right] \tag{3.14}$$

where $(1/24)A$ is a symmetric matrix with the eigenvalues

$$\{-5; -3.5; -2.5; -2; -1; -0.5; 2.5; 10\}$$

To these we add 5 and divide by 3, to get the canonical set

$$\{0; 1/2; 5/6; 1; 4/3; 3/2; 5/2; 5\} \tag{3.15}$$

with a characteristic equation leading to $e^{\mu\theta} = \sum_{j=1,3,\ldots}^{13} \mu^j S_j(\theta)$ where

$$\mu = \frac{M}{48} + \sigma = \left[\begin{array}{c|c} 0 & \frac{A}{72} + \frac{5}{3}E_8 \\ \hline -\frac{A}{72} - \frac{5}{3}E_8 & 0 \end{array}\right]; \qquad \sigma = \left[\begin{array}{c|c} 0 & \frac{5}{3}E_8 \\ \hline -\frac{5}{3}E_8 & 0 \end{array}\right]$$

$$e^{\sigma\theta} = \begin{bmatrix} \cos\frac{5}{3}\theta & \sin\frac{5}{3}\theta \\ -\sin\frac{5}{3}\theta & \cos\frac{5}{3}\theta \end{bmatrix} \otimes E_8 \tag{3.16}$$

and

$$\begin{aligned} S_1 &= \left(\theta - \frac{2{,}250{,}000}{144^2}\frac{\theta^{15}}{15!} + \cdots\right) \\ S_3 &= \left(\frac{\theta^3}{3!} - \frac{17{,}205{,}625}{144^2}\frac{\theta^{15}}{15!} + \cdots\right) \\ S_5 &= \left(\frac{\theta^5}{5!} - \frac{43{,}718{,}650}{144^2}\frac{\theta^{15}}{15!} + \cdots\right) \\ S_7 &= \left(\frac{\theta^7}{7!} - \frac{50{,}135{,}361}{144^2}\frac{\theta^{15}}{15!} + \cdots\right) \\ S_9 &= \left(\frac{\theta^9}{9!} - \frac{27{,}999{,}760}{144^2}\frac{\theta^{15}}{15!} + \cdots\right) \\ S_{11} &= \left(\frac{\theta^{11}}{11!} - \frac{7{,}378{,}528}{144^2}\frac{\theta^{15}}{15!} + \cdots\right) \\ S_{13} &= \left(\frac{\theta^{13}}{13!} - \frac{771{,}840}{144^2}\frac{\theta^{15}}{15!} + \cdots\right) \end{aligned} \tag{3.17}$$

Because of all the zeros, the determination of the coefficient a_{ij} of (2.7) is particularly simple for the odd nuclei and consists of inverting the matrix

$$B = \begin{bmatrix} \lambda_1 & \lambda_2 & 1 & \ldots & \lambda_n \\ \lambda_1^3 & \lambda_2^3 & 1 & \ldots & \lambda_n^3 \\ \cdot & \cdot & \cdot & & \cdot \\ \lambda_1^{2n-1} & \lambda_2^{2n-1} & 1 & \ldots & \lambda_n^{2n-1} \end{bmatrix} \tag{3.18}$$

Then because

$$\mu = \left[\begin{array}{c|c} 0 & C \\ \hline -C & 0 \end{array}\right]; \qquad \mu^3 = \left[\begin{array}{c|c} 0 & -C^3 \\ \hline C^3 & 0 \end{array}\right];$$

$$\mu^5 = \left[\begin{array}{c|c} 0 & C^5 \\ \hline -C^5 & 0 \end{array}\right], \ldots, \qquad C \equiv \frac{A}{72} + \frac{5}{3}E_8$$

the coefficients of $\sin\theta$, $\sin\lambda_2\theta, \ldots, \sin\lambda_n\theta$ in $e^{\mu\theta}$ are simply

$$B^{-1}\begin{bmatrix} c \\ c^3 \\ \vdots \\ c^{2n-1} \end{bmatrix} \tag{3.19}$$

where c is that element X_4 in the middle of the center column of A (labeled by $[\Lambda]$) or an adjacent state X_5, and c^3 is chosen from C^3, etc.

Finally, from (3.16) the harmonics of $e^{(M/4\theta)\theta}$ are

$$64X_4 = +\left(9\sin\frac{\theta}{2} + 5\sin\frac{5}{6}\theta + \frac{3}{2}\sin\theta + \frac{1}{2}\sin\frac{4}{3}\theta + 3\sin\frac{3}{2}\theta + 15\sin\frac{5}{2}\theta + 22.5\sin 5\theta\right)\sin\frac{5}{3}\theta \tag{3.20a}$$

$$64X_5 = \left(9\sin\frac{\theta}{2} + 5\sin\frac{5}{6}\theta - \frac{3}{2}\sin\theta - \frac{1}{2}\sin\frac{4}{3}\theta + 3\sin\frac{3}{2}\theta + 15\sin\frac{5}{2}\theta - \frac{45}{2}\sin 5\theta\right)\sin\frac{5}{3}\theta \tag{3.20b}$$

We find that $X_5(\theta) = -X_4(1080^\circ - \theta)$ is the complement of $X_4(\theta)$ and if we label all eight elements of the center column in numerical order beginning with X_1, then we obtain the remaining anticomplementary relationships

$$X_8(\theta) = -X_1(1080^\circ - \theta), \qquad X_7(\theta) = -X_2(1080^\circ - \theta)$$

$$X_6(\theta) = -X_3(1080^\circ - \theta)$$

where only the signs of the third, fourth, and seventh terms differ in each pair! Moreover, it is easily verified that $dX_i/d\theta|_{\theta=0}$, $i = 1, \ldots, 8$, are in fact the elements of the relevant column of μ.

Returning to (3.20), we can use multiple angle formulas to express $\sin 5\theta$ in terms of $\sin\theta$, $\{\sin\frac{5}{2}\theta;\ \sin\frac{3}{2}\theta\}$ in terms of $\sin(\theta/2)$, and $\sin(8\theta/6)$ in terms of $\sin\frac{5}{6}\theta$, so that there are actually three different sets of translated angular momentum, of which only the second has been measured. In the absence of any stronger indication we shall take X_4 and X_5 as the eigenfunctions of the eigenvalues $\frac{1}{2}$ and $\frac{3}{2}$ and these eigenfunctions are plotted in Fig. 2. There is a shell structure in accord with the measurements of Blank *et al.* (1991), but it is remarkable that mesons pass from the inner shell to the outer one and if the same pattern is followed for ^{11}Li, two extra

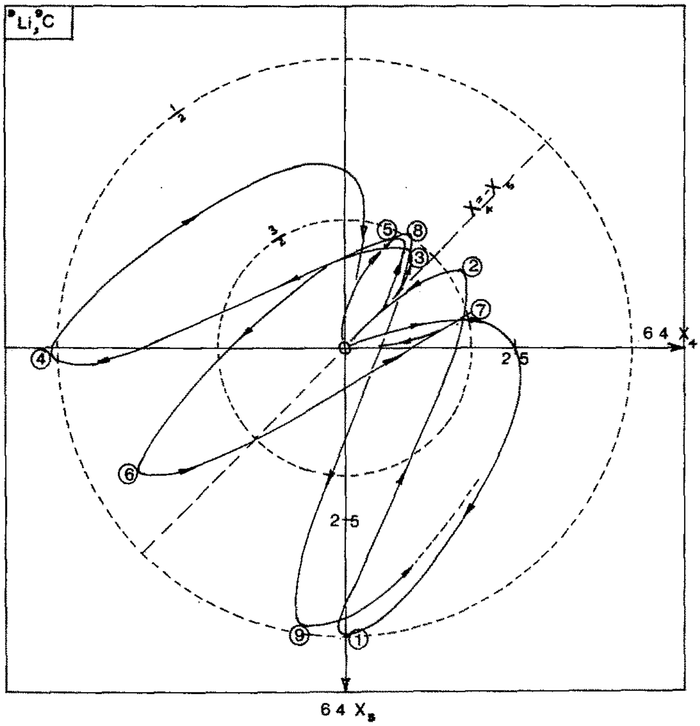

Fig. 2. Geodesics on the manifold of ^{9}Li.

nucleons could be loosely bound to the $j = \frac{1}{2}$ core and account for the nucleon halo. Unfortunately *CP* symmetry is broken for this case, so that direct calculation is not possible, but because $X_5 = -X_4(1080° - \theta)$ there is another picture symmetric about $X_4 = -X_5$.

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REFERENCES

Arnold, V. I. (1989). *Mathematical Methods of Classical Mechanics*, 2nd ed., Springer-Verlag, New York.

Barut, A. O., and Raczka, R. (1977). *Theory of Group Representations and Applications*, PWN, Warsaw, Chapter 9.
Biedenharn, L. C., and Louck, J. D. (1981). *Angular Momentum in Quantum Physics*, Addison-Wesley, Reading, Massachusetts.
Blank, B., *et al.* (1991). *Zeitschrift für Physik A Hadrons and Nuclei*, **340**, 41.
Boerner, H. (1963). *Representations of Groups*, North-Holland, Amsterdam, p. 299.
de Wet, J. A. (1971). *Proceedings of the Cambridge Philosophical Society*, **70**, 485.
de Wet, J. A. (1973). *Proceedings of the Cambridge Philosophical Society*, **74**, 149.
de Wet, J. A. (1987). *Foundations of Physics*, **17**, 993.
Gilmore, R., and Draayer, J. P. (1985). *Journal of Mathematical Physics*, **26**, 3053.
Kemmer, N. (1943). *Proceedings of the Cambridge Philosophical Society*, **39**, 189.
Kobayashi, S., and Nomizu, K. (1969). *Foundations of Differential Geometry*, Wiley-Interscience, New York, Vol. II, Chapter XI.10.
Wong Yung-Chow (1967). *Proceedings of the National Academy of Science*, **57**, 589.